Conditional probability cont., Independence

Modeling (lack of) causality

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What we aim to model

E.g. ① ②

H T

T H

$X = \{\text{flip 2 coins}\}$

①

	H_1	T_1
H_2	0.25	0.25
T_2	0.25	0.25

$\mathbb{P}(H_1 \cap H_2) = \mathbb{P}(H_1)\mathbb{P}(H_2)$

$\mathbb{P}(H_1 \cap T_2) = \mathbb{P}(H_1)\mathbb{P}(T_2)$

...

②

	H_1	T_1
H_2	0	0.5
T_2	0.5	0

$\mathbb{P}(H_1 \cap H_2) \overset{?}{=} \mathbb{P}(H_1)\mathbb{P}(H_2)$

$0 \neq 0.5 \cdot 0.5 = 0.25$

Independence

Definition. *Given a random variable* $X = (\Omega, \mathbb{P})$*, two events* $A, B \subset \Omega$ *are said to be independent if the following equation holds:*

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B).$$

$X = \{2$ weighted coins (indep.), each w/ $\mathbb{P}(H) = 0.7\}$

	H	T
H	0.7^2	$0.7 \cdot 0.3$
T	$0.7 \cdot 0.3$	0.3^2

$\mathbb{P}(C_1)$ $\mathbb{P}(C_2)$

$$\begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} c & d \end{pmatrix}$$

Equivalent definition

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$$

$$\iff \mathbb{P}(A|B)\mathbb{P}(B) = \mathbb{P}(A)\mathbb{P}(B) \iff \boxed{\mathbb{P}(A|B) = \mathbb{P}(A)}$$

$$\iff \mathbb{P}(B|A)\mathbb{P}(A) = \mathbb{P}(A)\mathbb{P}(B) \iff \boxed{\mathbb{P}(B|A) = \mathbb{P}(B)}$$

Examples

E.g., die, $\Omega = \{1,2,3,4,5,6\}$ $\mathbb{P}$ = uniform

① $A = \{1,3,5\}$ $B = \{2,4,6\}$

$\mathbb{P}(A \cap B) = \mathbb{P}(\emptyset) = 0$

$\mathbb{P}(A) = \frac{1}{2}$, $\mathbb{P}(B) = \frac{1}{2} \Rightarrow \mathbb{P}(A)\mathbb{P}(B) = 0.25$

$\neq 0$

② $A = \{1,2\}$, $B = \{1,3\}$

$\mathbb{P}(A \cap B) = \mathbb{P}(\{1\}) = \frac{1}{6}$

$\mathbb{P}(A) = \frac{1}{3}$, $\mathbb{P}(B) = \frac{1}{3}$, $\frac{1}{6} \neq \frac{1}{9}$

③ $\Omega = \{1,2,3,4\}$

$\mathbb{P}(A \cap B) = \mathbb{P}(\{1\}) = \frac{1}{4}$

$\mathbb{P}(A) = \frac{1}{2}$, $\mathbb{P}(B) = \frac{1}{2}$, $\Rightarrow \mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$

Independence in joint distributions

	H_1	T_1
H_2	a	b
T_2	c	d

$$\mathbb{P}(H_1 \cap H_2) = \mathbb{P}(H_1)\,\mathbb{P}(H_2)$$

$$a = (a+c)(a+b)$$

$$= \boxed{\begin{pmatrix} a' \\ b' \end{pmatrix} \begin{pmatrix} c' & d' \end{pmatrix}} := \begin{pmatrix} a'c' & a'd' \\ b'c' & b'd' \end{pmatrix}$$

Mutual independence

$\prod_{i=1}^{n} a_i := a_1 \cdot a_2 \cdot \dots \cdot a_n$

$\prod_{i \in I} a_i$

Definition. *Given a random variable* $X = (\Omega, \mathbb{P})$, *a collection of events* $A_1, \dots, A_n \subset \Omega$ *are said to be mutually independent if the following equation holds for every subset* $I \subset \{1, \dots, n\}$ *such that* $|I| \geq 2$:

$$\mathbb{P}(\cap_{i \in I} A_i) = \Pi_{i \in I} \mathbb{P}(A_i).$$

$n=3$

$$\left.\begin{aligned} \mathbb{P}(A_1 \cap A_3) &= \mathbb{P}(A_1)\mathbb{P}(A_3); \\ \mathbb{P}(A_2 \cap A_3) &= \mathbb{P}(A_2)\mathbb{P}(A_3) \\ \mathbb{P}(A_1 \cap A_2) &= \mathbb{P}(A_1)\mathbb{P}(A_2) \end{aligned}\right\} \quad \mathbb{P}(A_1 \cap A_2 \cap A_3) = \mathbb{P}(A_1)\mathbb{P}(A_2)\mathbb{P}(A_3)$$

Why mutual independence?

$\Omega = \{H \times T\}^{(1)} \times \{H \vee T\}^{(2)}$

$\mathbb{P}(A \cap B \cap C) = \frac{1}{4}$

$A = \{1^{st} \text{ coin heads}\} = \{(H,H),(H,T)\}$

$B = \{2^{nd} \text{ coin heads}\}$

$C = \{\text{Both coins same}\} = \{(H,H),(T,T)\}$

A ind w/ B, B ind w/ C, A ind. w/ C

but, A, B, C mutually indep.

General product rule

Theorem. *Given a random variable* $X = (\Omega, \mathbb{P})$, *and a collection of events* $A_1, \ldots, A_n \subset \Omega$, *the following equation holds:*

$$\mathbb{P}(A_1 \cap A_2 \cap \cdots \cap A_n) = \mathbb{P}(A_1)\mathbb{P}(A_2 \mid A_1)\mathbb{P}(A_3 \mid A_1 \cap A_2) \cdots \mathbb{P}(A_n \mid A_1 \cap \cdots \cap A_{n-1}).$$

General product rule

Proof Induction, Base case, $n=2$. $\mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1)\,\mathbb{P}(A_2 | A_1)$.

By definition, holds. Assume prop. holds for $n-1$.

$$\mathbb{P}(A_1 \cap A_2 \cap \dots \cap A_n) = \mathbb{P}((A_1 \cap \dots \cap A_{n-1}) \cap A_n)$$

$$= \mathbb{P}(A_1 \cap \dots \cap A_{n-1})\,\mathbb{P}(A_n | A_1 \cap A_2 \cap \dots \cap A_{n-1})$$

$$= \mathbb{P}(A_1)\,\mathbb{P}(A_2 | A_1) \cdots \mathbb{P}(A_n | A_1 \cap \dots \cap A_{n-1})$$

$\boxed{}$

General product rule: equivalent forms

$$P(A_1 \cap A_2 \cap A_3) = P(A_2 \cap A_1 \cap A_3)$$

$$= \underline{P(A_2)}\, P(A_1 | A_2)\, P(A_3 | A_1 \cap A_2)$$

$$P(A_1) P(A_2 | A_1) \ldots$$

$$P(\text{win, switch}) = P\big((P_1 \cap C_2 \cap H_3) \cup (P_1 \cap C_3 \cap H_2) \cup \ldots\big)$$

$$P(P_1 \cap C_2 \cap H_3) + \ldots,$$

$$= P(P_1 \cap \overline{C_1}) + P(P_2 \cap \overline{C_2}) + P(P_3 \cap \overline{C_3})$$

car door

Monty Hall revisited

host's pick

person's door choice

$$P_1,\ P_2,\ P_3 \qquad C_1,\ C_2,\ C_3 \qquad H_1,\ H_2,\ H_3$$

$$\mathbb{P}(\text{win}) = \mathbb{P}\big((P_1 \cap C_1) \cup (P_2 \cap C_2) \cup (P_3 \cap C_3)\big)$$

$$\mathbb{P}(P_1 \cap C_1) = \mathbb{P}(P_1 \cap C_1 \cap H_1) + \underline{\mathbb{P}(P_1 \cap C_1 \cap H_2)} + \mathbb{P}(P_1 \cap C_1 \cap H_3)$$

$$\underline{\mathbb{P}(H_1 \cap C_2 \cap P_3)} = \mathbb{P}(C_2)\,\mathbb{P}(P_3 \mid C_2)\,\mathbb{P}(H_1 \mid C_2 \cap P_3)$$

$$\frac{1}{3} \cdot \frac{1}{3} \cdot 1 = \boxed{\frac{1}{9}}$$